# Lecture 33. The Gram-Schmidt process

**Thm** (The Gram-Schmidt process)

If $V$ is a subspace of $\mathbb{R}^n$ together with a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$, it has an orthogonal basis given as follows:

- $\vec{u}_1 = \vec{v}_1$,
- $\vec{u}_2 = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1$,
- $\vec{u}_3 = \vec{v}_3 - \dfrac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \dfrac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$,

$$\vdots$$

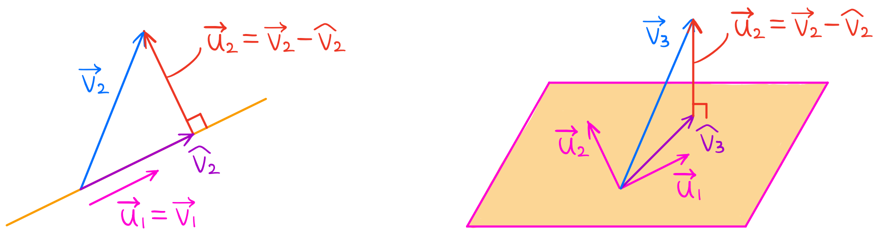
- $\vec{u}_m = \vec{v}_m - \dfrac{\vec{v}_m \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \dfrac{\vec{v}_m \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 - \cdots - \dfrac{\vec{v}_m \cdot \vec{u}_{m-1}}{\vec{u}_{m-1} \cdot \vec{u}_{m-1}} \vec{u}_{m-1}$

**Note** (1) We can replace each $\vec{u}_i$ by its multiple to simplify computation.

(2) The orthogonal basis above is recursively constructed by

$$\vec{u}_i = \vec{v}_i - \hat{v}_i$$

where $\hat{v}_i$ is the orthogonal projection of $\vec{v}_i$ onto the subspace spanned by $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_{i-1}$.




$\Rightarrow$ $\vec{u}_i$ is orthogonal to $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_{i-1}$.

Ex Find an orthogonal basis of each vector space.

(1) The column space of

$$A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 3 & 4 & -1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Sol RREF(A) has leading 1's in column 1 and column 2.

$\Rightarrow$ A basis of Col(A) is given by

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{(column 1 and column 2 in A)}$$

We apply the Gram-Schmidt process.

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \vec{v}_2 - \frac{6}{6}\vec{u}_1 = \vec{v}_2 - \vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

$\Rightarrow$ Col(A) has an orthogonal basis given by $\boxed{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}$

(2) The null space of

$$B = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 2 & 3 \end{bmatrix} \text{ with } \mathrm{RREF}(B) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

<u>Sol</u> We solve the equation $B\vec{x} = \vec{0}$ using RREF(B).

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 + 3x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = -x_3 - 3x_4 \end{cases} \underset{\substack{x_3 = s \\ x_4 = t}}{\Longrightarrow} \vec{x} = s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ A basis of Nul(B) is given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The Gram-Schmidt process yields

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \vec{v}_2 - \frac{3}{3}\vec{u}_1 = \vec{v}_2 - \vec{u}_1 = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}.$$

$\Rightarrow$ Nul(B) has an orthogonal basis given by $\boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}}$

(3) The row space of

$$C = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \quad \text{with} \quad \text{RREF}(C) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Sol A basis of Row(C) is given by the nonzero rows in RREF(C)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

The Gram-Schmidt process yields

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = \vec{v}_2 + \frac{1}{2} \vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \vec{u}_2' = 2\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2' \cdot \vec{u}_2'} \vec{u}_2' = \vec{v}_3 + \frac{3}{2} \vec{u}_1 - \frac{1}{2} \vec{u}_2' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

$\Longrightarrow$ Row(C) has an orthogonal basis given by $\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}}$